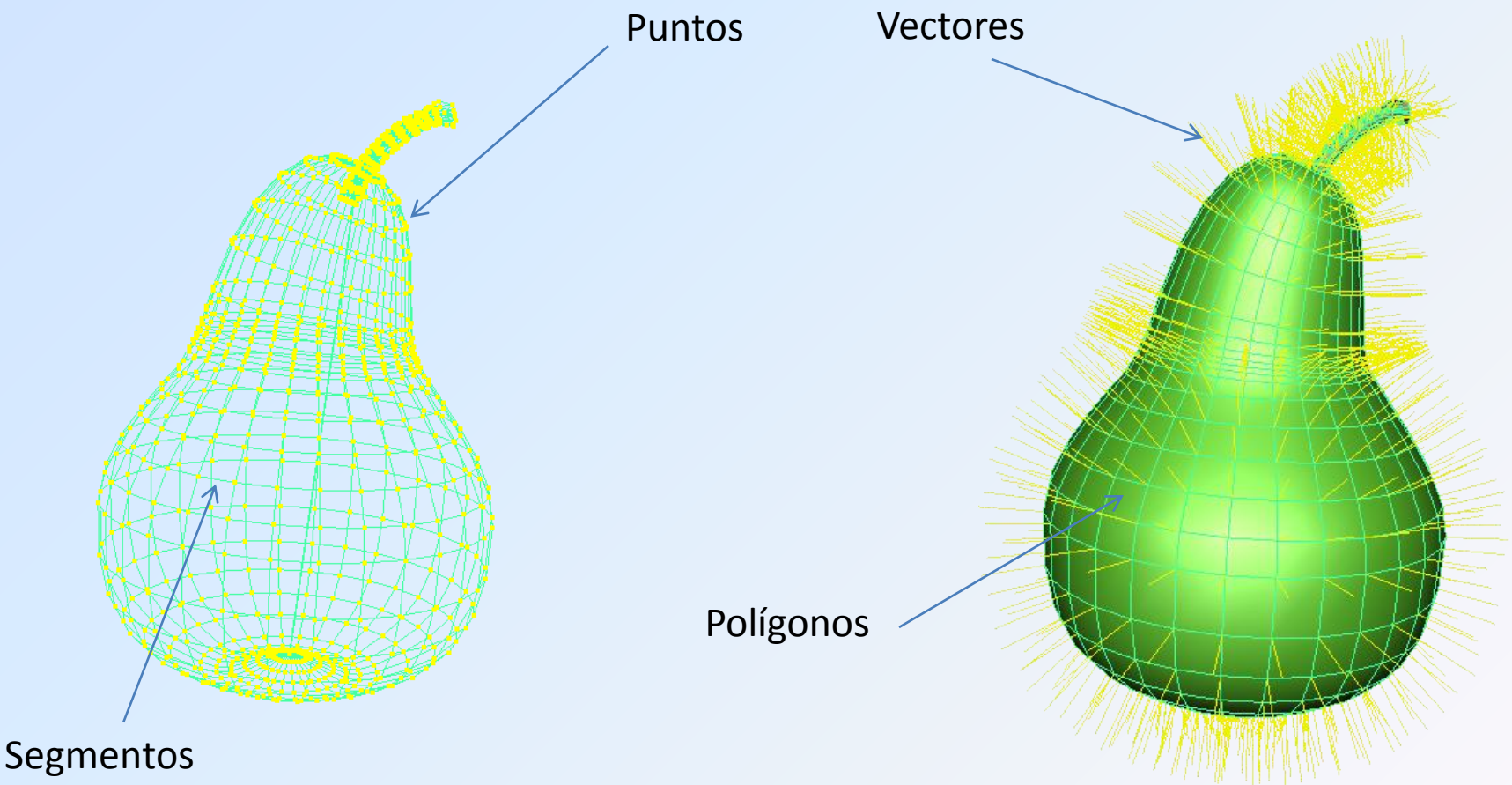
Introducción al Algebra Geométrica para la computación gráfica

Presentado por el Lic. Eduardo Roa

Algebra Vectorial



Algebra Vectorial

Producto Escalar

$$a.b = a_1b_1 + a_2b_2 + a_3b_3$$

$$a.b = \|a\| \, \|b\| \, cos(\alpha)$$

Información del ángulo entre los vectores

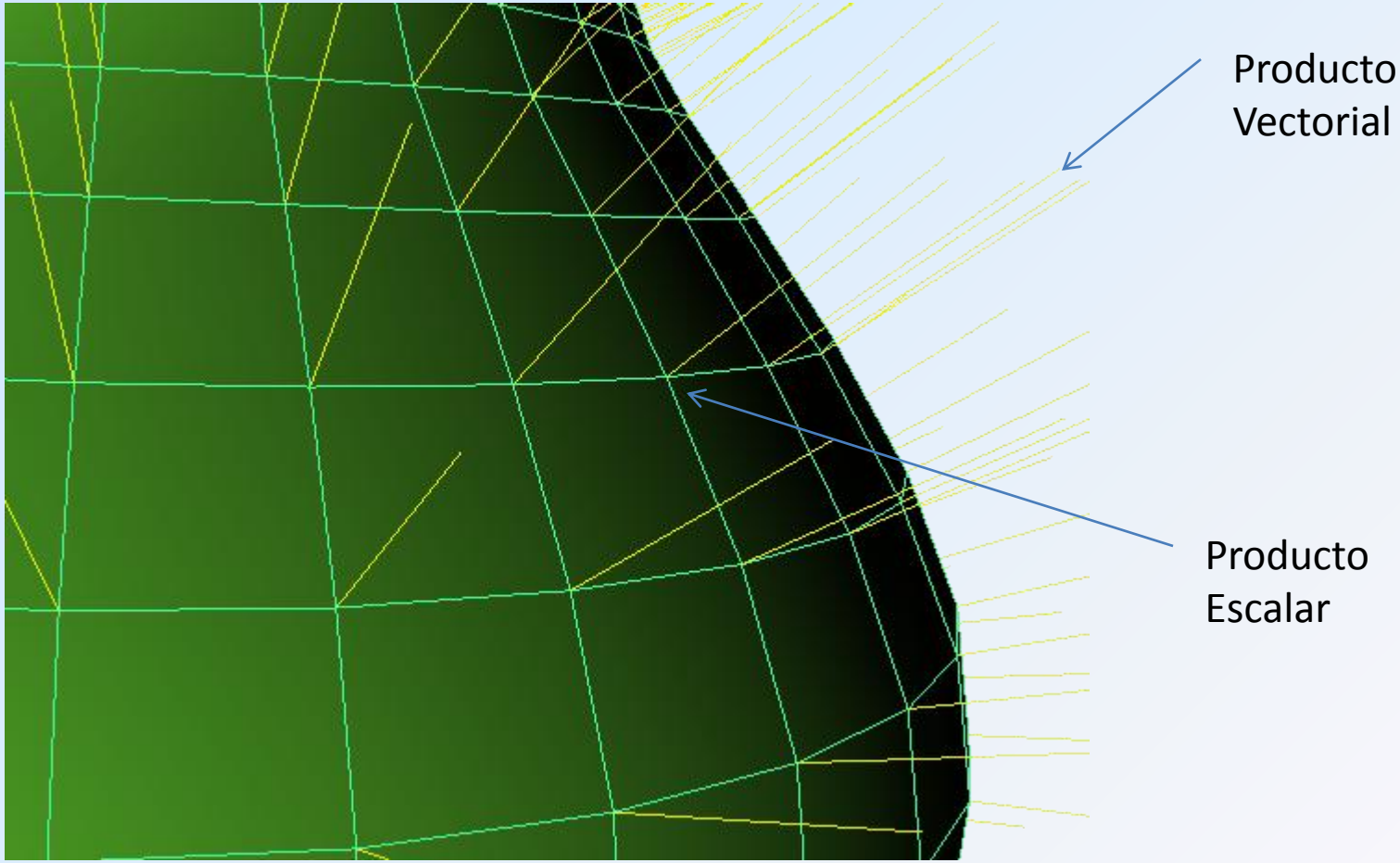
Producto Vectorial

$$a \times b = (a_2b_3 - b_2a_3)i + (a_3b_1 - b_3a_1)j + (a_1b_2 - b_2a_1)k$$

$$\|a \times b\| = \|a\| \, \|b\| \, sin\alpha$$

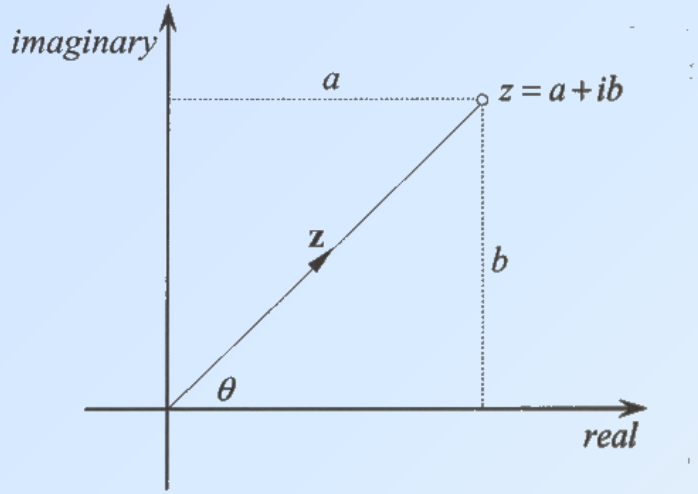
Vector perpendicular y área del paralelogramo

Algebra Vectorial

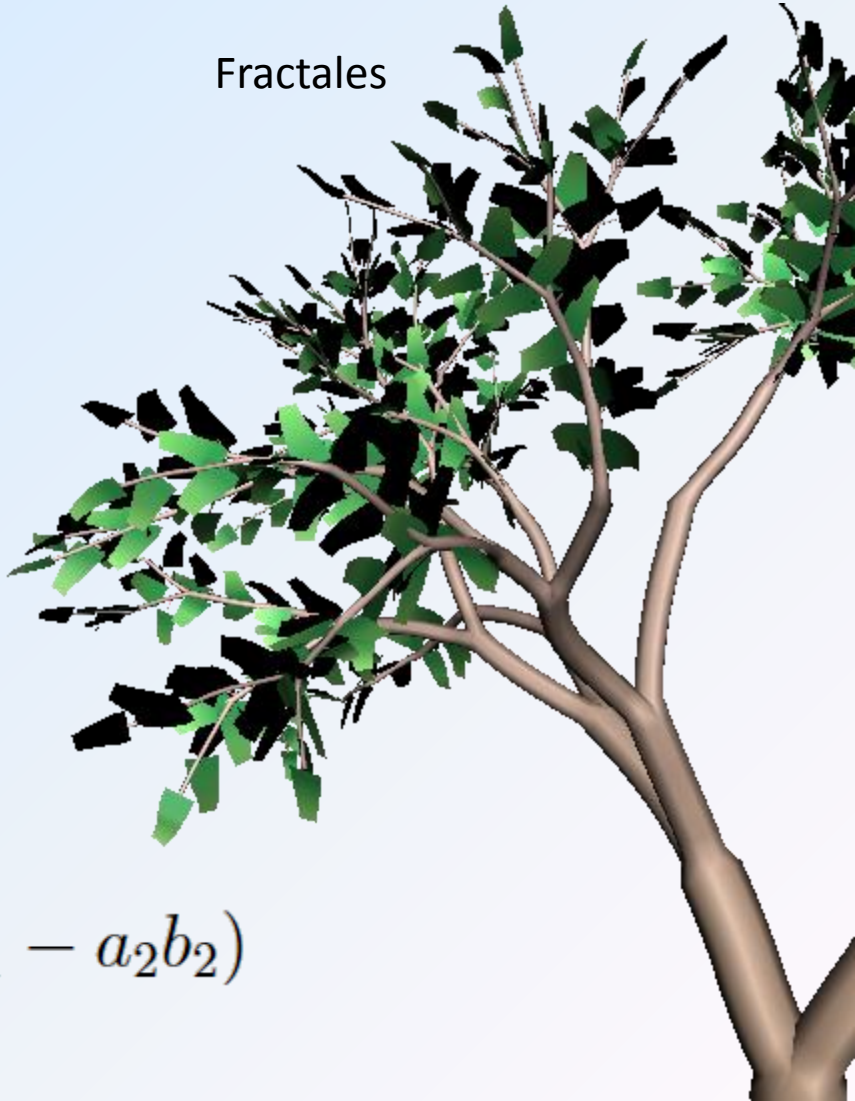


Aplicación en iluminación

Números Complejos



Fractales

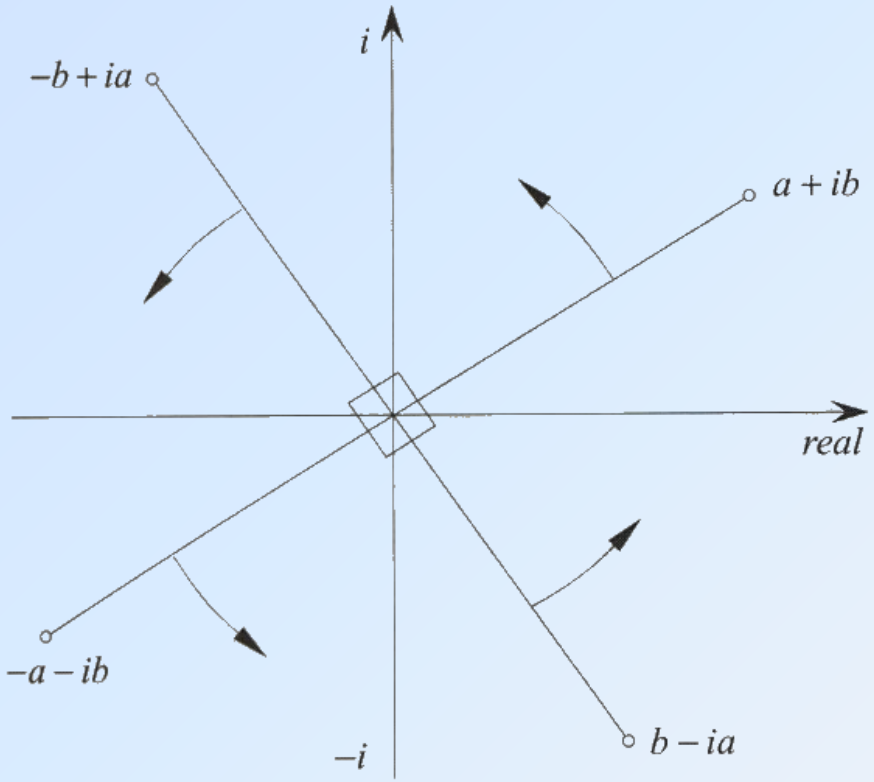


$$z = a + ib$$

$$z^* = a - ib$$

$$z_1 z_2^* = (a_1 a_2 + b_1 b_2) + i(a_2 b_1 - a_2 b_2)$$

Números Complejos



Rotaciones en los complejos

$$z = r(cos\phi + isen\phi)$$
$$= re^{i\phi}$$

$$z' = ze^{i\varphi}$$
$$= re^{i\phi}e^{i\varphi}$$
$$= re^{i\phi+\varphi}$$

Quaterniones

Quaternion

$$q = \alpha + x_1 i + y_1 j + z_1 k$$

$$q = \alpha + v_1$$
$$v_1 = (x_1, y_1, z_1)$$

$$i^2 = j^2 = k^2 = ijk = -1$$
$$ii = k \quad jk = i \quad ki = j \quad ji = -k \quad kj = -i \quad ik = -j$$

Inversa de un quaternion

$$q^{-1} = \frac{\alpha + x_1^2 + y_1^2 + z_1^2}{\| q \|}$$

Tomb Raider, citado como el primer juego distribuido en masa que uso quaterniones para lograr rotaciones suaves en 3D

Quaterniones

Rotaciones

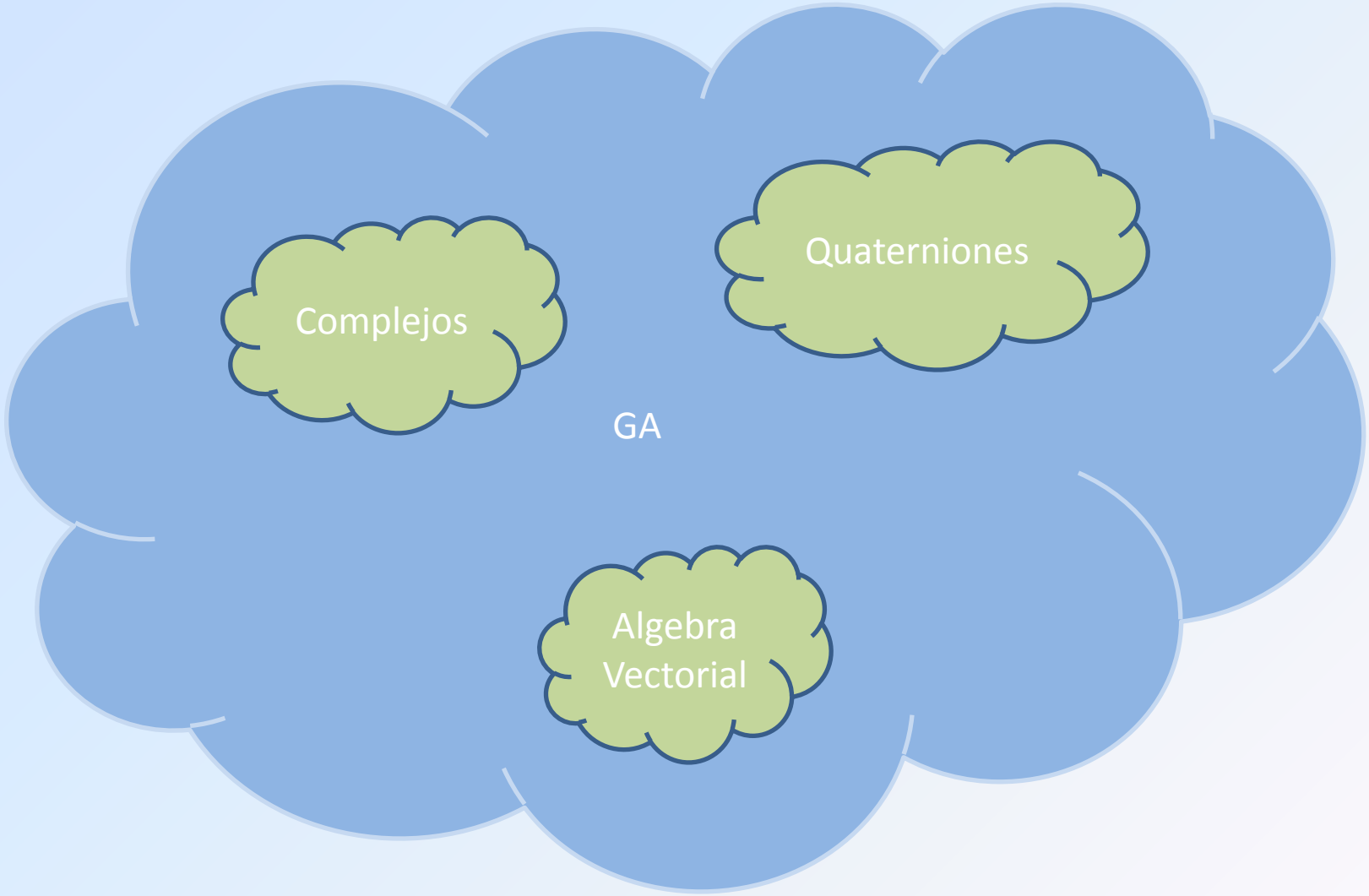
$$p' = qpq^{-1}$$

$$p = xi + yj + zk$$
$$q = cos(\sigma/2) + sen(\sigma/2)\widehat{u}$$
$$q^{-1} = cos(\sigma/2) - sen(\sigma/2)\widehat{u}$$

Donde el eje de rotación es:

$$\widehat{u} = x_u i + y_u j + z_u k \qquad (\| \widehat{u} \| = 1)$$

Algebra Geométrica



Algebra Geométrica

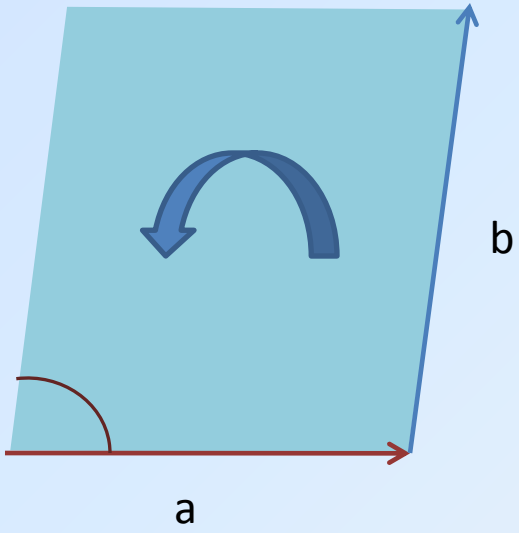
$a \wedge b$ → Bivector → Describe la orientacion de un plano en termino de dos vectores, y su magnitud es el area del paralelogramo formado por los mismos

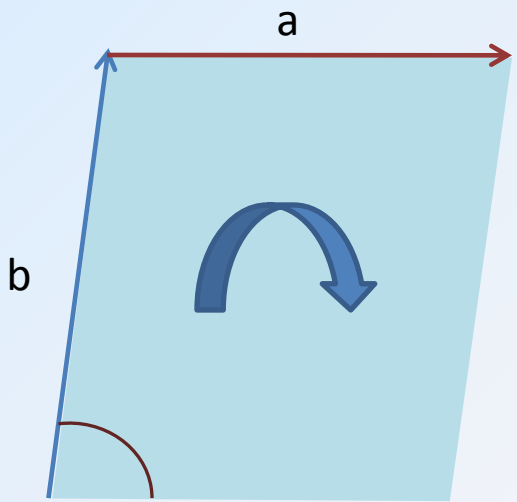
$$\| a \wedge b \| = \| a \| \| b \| sen\theta$$

- $a \wedge b = -b \wedge a$
- $a \wedge (b + c) = a \wedge b + a \wedge c$
- $\|a \wedge a\| = \|a\|\|a\| sin0 = 0$

Algebra Geométrica

Orientación?





$$a \wedge b$$

$$b \wedge a$$

Algebra Geométrica

Sean dos vectores en R^2

$$a = a_1 e_1 + a_2 e_2$$
$$b = b_1 e_1 + b_2 e_2$$

$$\begin{aligned} a \wedge b &= (a_1 e_1 + a_2 e_2) \wedge (b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 (e_1 \wedge e_1) + a_1 b_2 (e_1 \wedge e_2) + a_2 b_1 (e_2 \wedge e_1) + a_2 b_2 (e_2 \wedge e_2) \end{aligned}$$

$$e_1 \wedge e_1 = e_2 \wedge e_2 = 0 \quad y \quad e_2 \wedge e_1 = -e_1 \wedge e_2$$

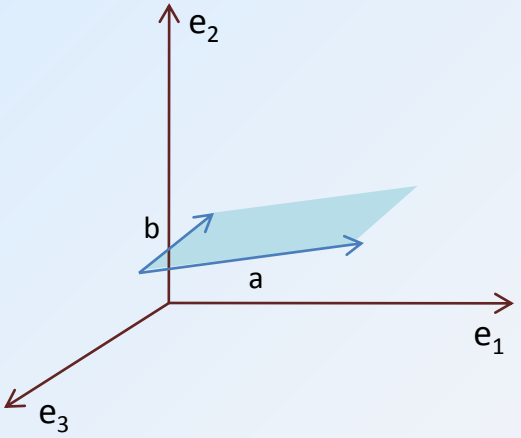
$$a \wedge b = (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2)$$

Área

Algebra Geométrica

Sean dos vectores en R^3

$$a = a_1e_1 + a_2e_2 + a_3e_3$$
$$b = b_1e_1 + b_2e_2 + b_3e_3$$

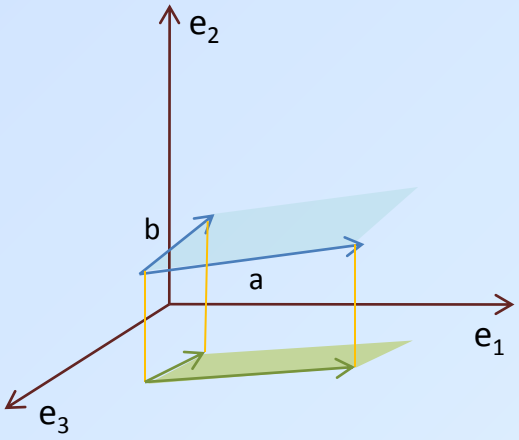


$$e_1 \wedge e_1 = e_2 \wedge e_2 = e_3 \wedge e_3 = 0$$

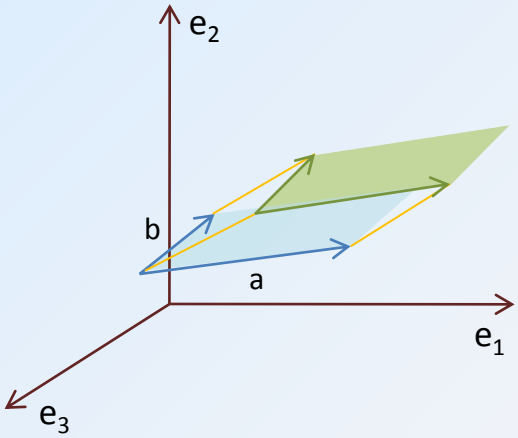
$$e_2 \wedge e_1 = -e_1 \wedge e_2 \quad e_1 \wedge e_3 = -e_3 \wedge e_1 \quad e_3 \wedge e_2 = -e_2 \wedge e_3$$

$$a \wedge b = (a_1b_2 - a_2b_1)(e_1 \wedge e_2) + (a_2b_3 - a_3b_2)(e_2 \wedge e_3) + \\ + (a_3b_1 - a_1b_3)(e_3 \wedge e_1)$$

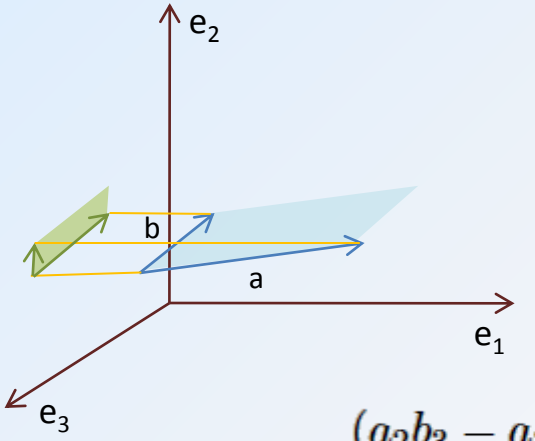
Algebra Geométrica



$$(a_3b_1 - a_1b_3)(e_3 \wedge e_1)$$



$$(a_1b_2 - a_2b_1)(e_1 \wedge e_2)$$



$$(a_2b_3 - a_3b_2)(e_2 \wedge e_3)$$

Algebra Geométrica

Relación de las áreas

$$\|a \wedge b\|^2 = (a_1 b_2 - a_2 b_1)^2 + (a_2 b_3 - a_3 b_2)^2 + (a_3 b_1 - a_1 b_3)^2$$

Sean dos vectores en R^4

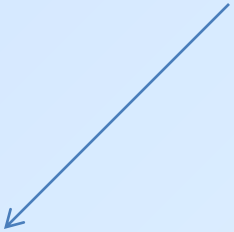
$$\begin{aligned} a \wedge b = {} & (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2) + (a_2 b_3 - a_3 b_2)(e_2 \wedge e_3) \\ & + (a_3 b_1 - a_1 b_3)(e_3 \wedge e_1) + (a_1 b_4 - a_4 b_1)(e_1 \wedge e_4) \\ & + (a_2 b_4 - a_4 b_2)(e_2 \wedge e_4) + (a_3 b_4 - a_4 b_3)(e_3 \wedge e_4) \end{aligned}$$

$$C_{n,2} = \frac{n!}{(n-2)!2!}$$

Algebra Geométrica

Producto Geométrico

$$ab = a.b + a \wedge b$$



- $a(bc) = (ab)c = abc$
- $a(b+c) = ab + ac$
- $(b+c)a = ba + ca$
- $a^2 = \pm \|a\|^2$

$$a \wedge b = \frac{1}{2}(ab - ba)$$

$$a.b = \frac{1}{2}(ab + ba)$$

Algebra Geométrica

Producto Geométrico de los vectores base

$$e_1 e_1 = e_1 . e_1 + e_1 \wedge e_1$$

$$e_1 e_1 = 1 + 0$$

$$e_1 e_1 = e_1^2 = 1$$

$$e_1 e_2 = e_{12} = e_1 . e_2 + e_1 \wedge e_2$$

$$e_{12} = 0 + e_1 \wedge e_2$$

$$e_{12} = e_1 \wedge e_2$$

$$e_2 e_1 = e_{21} = e_2 . e_1 + e_2 \wedge e_1$$

$$e_{21} = -e_1 \wedge e_2 = -e_{12}$$

Algebra Geométrica

Sean dos vectores en R^2

$$a = a_1 e_1 + a_2 e_2$$
$$b = b_1 e_1 + b_2 e_2$$

$$ab = (a_1 e_1 + a_2 e_2).(b_1 e_1 + b_2 e_2) + a \wedge b$$

$$ab = (a_1 b_1 + a_2 b_2) + a \wedge b$$

$$ab = (a_1 b_1 + a_2 b_2) + (a_1 b_2 - a_2 b_1)(e_1 \wedge e_2)$$

Algebra Geométrica

$$(e_1 \wedge e_2)^2 = (e_1 \wedge e_2)(e_1 \wedge e_2) = e_1 e_2 e_1 e_2$$

$$(e_1 \wedge e_2)^2 = -e_1 e_1 e_2 e_2$$

$$(e_1 \wedge e_2)^2 = -e_1^2 e_2^2$$

$$(e_1 \wedge e_2)^2 = -1$$

$$(e_2 \wedge e_3)^2 = -1$$

$$(e_3 \wedge e_1)^2 = -1$$

Algebra Geométrica

I como rotor

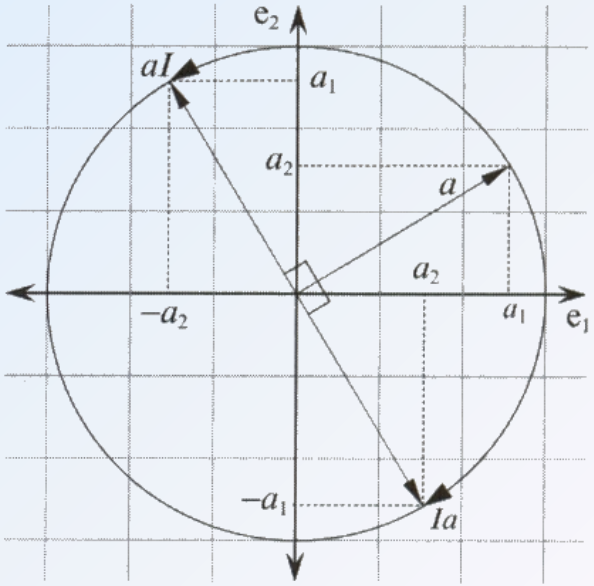
$$a = a_1 e_1 + a_2 e_2$$

$$aI = ae_1 e_2 = (a_1 e_1 + a_2 e_2) e_1 e_2 = a_1 e_1^2 e_2 + a_2 e_2 e_1 e_2$$

$$aI = a_1 e_2 - a_2 e_2^2 e_1$$

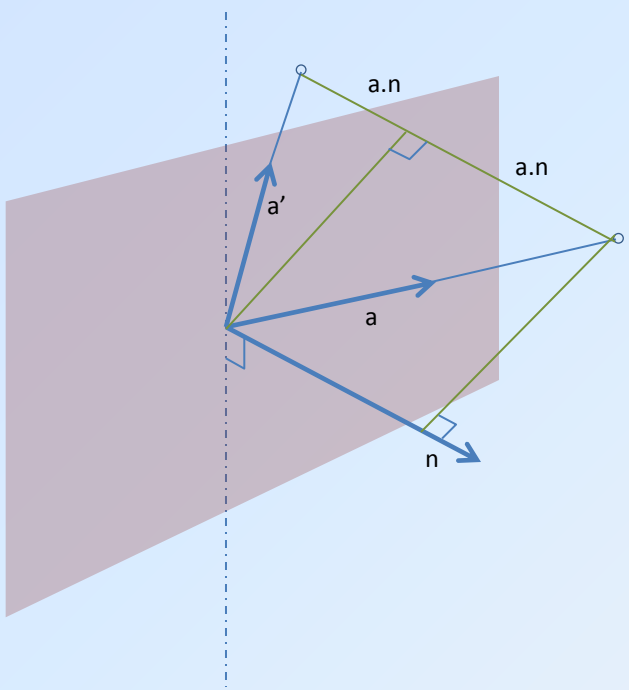
$$aI = -a_2 e_1 + a_1 e_2$$

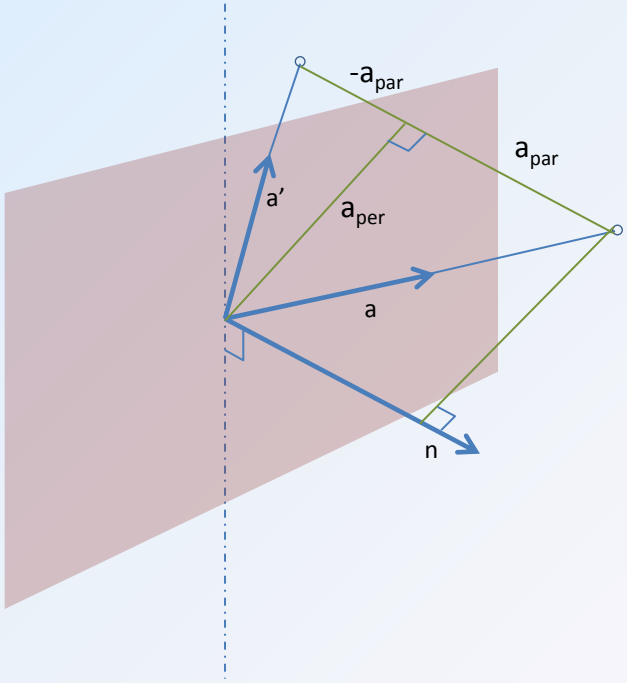
$$Ia = a_2 e_1 - a_1 e_2$$



Algebra Geométrica

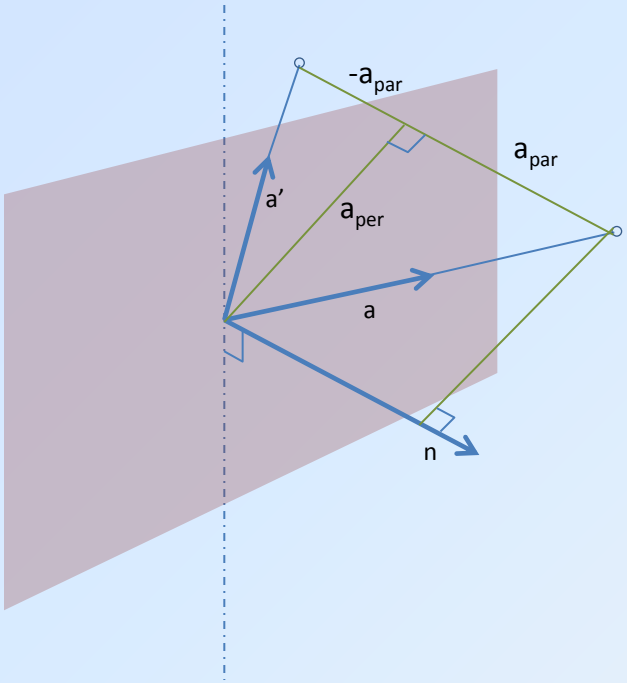
Reflexiones





$$a' = a - (2a.n)n$$

Algebra Geométrica



n es un vector unitario

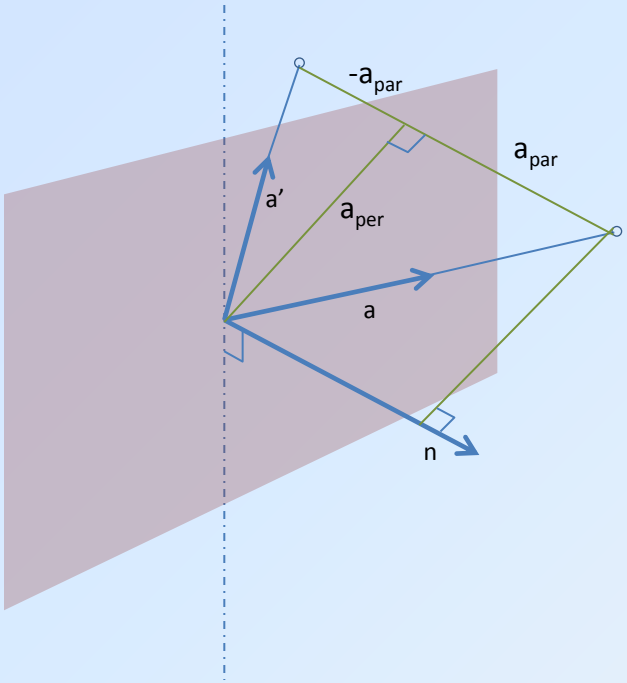
$$a = n^2 a = n(na)$$

$$a = n(n.a + n \wedge a)$$

$$a = a_{per} + a_{par}$$

$$a_{per} = n(n \wedge a) \qquad a_{par} = (n.a)n$$

Algebra Geométrica



$$a' = a_{per} - a_{par}$$

$$a' = n(n \wedge a) - (n.a)n$$

$$a' = -(n.a)n + n(n \wedge a)$$

Pero aB = -Ba

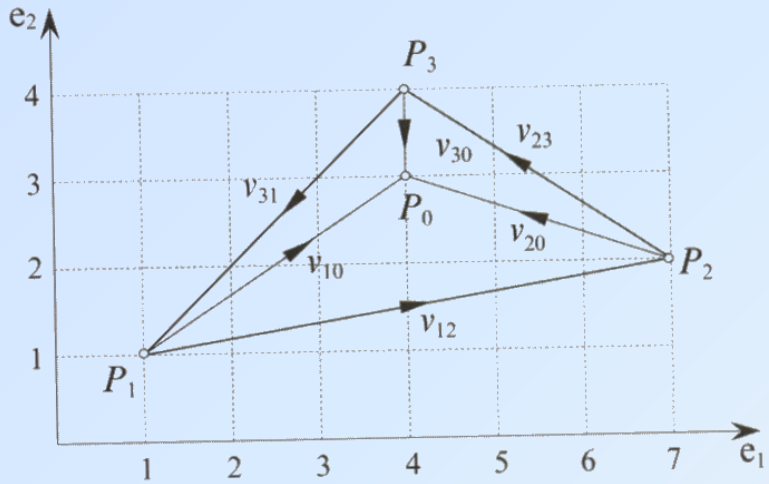
$$a' = -(n.a)n - (n \wedge a)n$$

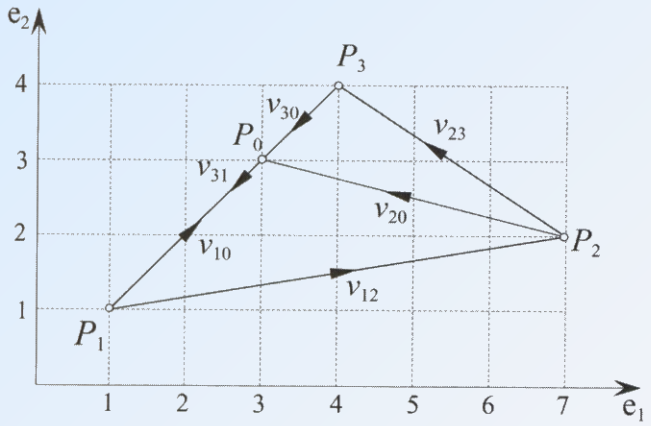
$$a' = -(n.a + n \wedge a)n$$

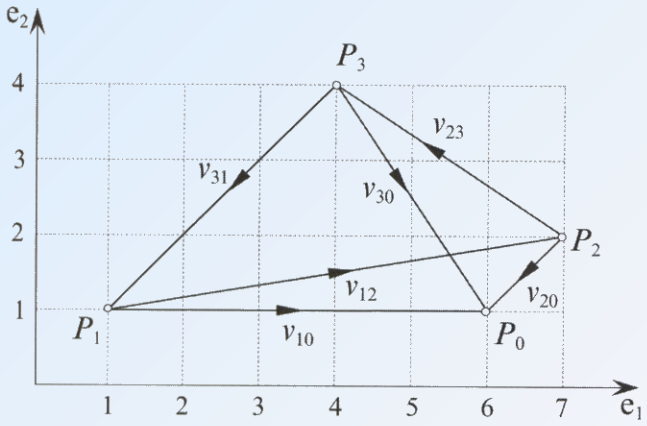
$$a' = -nan$$

Algebra Geométrica

Punto dentro de un triangulo



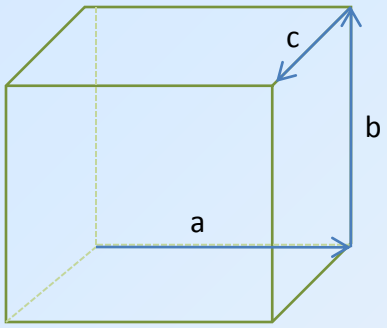




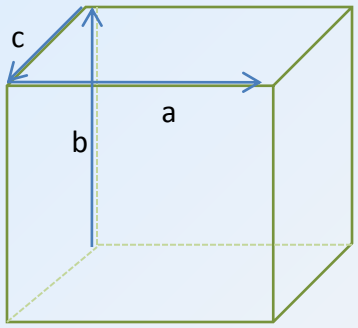
Algebra Geométrica

Trivectores $a \wedge b \wedge c$ ⟶ Volúmenes

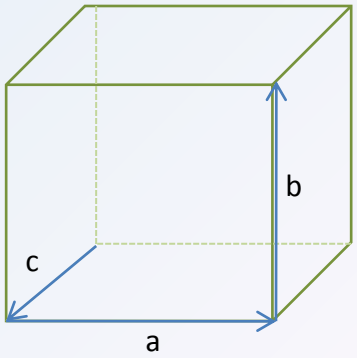
$$a \wedge b \wedge c = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} e_1 \wedge e_2 \wedge e_3$$



$(a \wedge b) \wedge c$



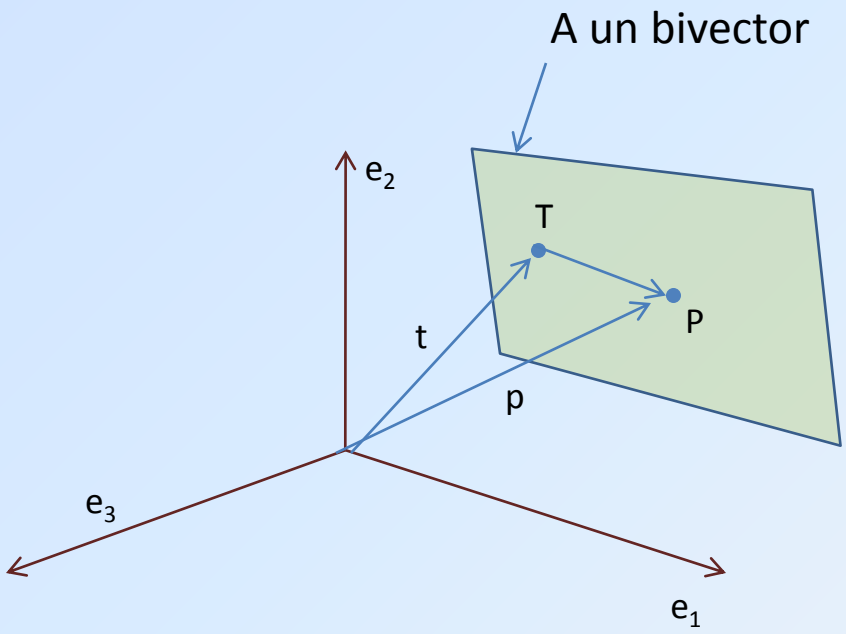
$(b \wedge c) \wedge a$



$(c \wedge a) \wedge b$

Algebra Geométrica

Orientación de un punto y un plano



$$v = p - t$$

$$A \wedge v = A \wedge (p - t)$$

Como A y v son paralelos:

$$A \wedge (p - t) = 0$$

Sea q en el espacio:

$$A \wedge (q - t) = 0$$

$$A \wedge (q - t) > 0$$

$$A \wedge (q - t) < 0$$

Algebra Geométrica

Inversa de un vector

$$B = ab$$

$$Bb = abb$$

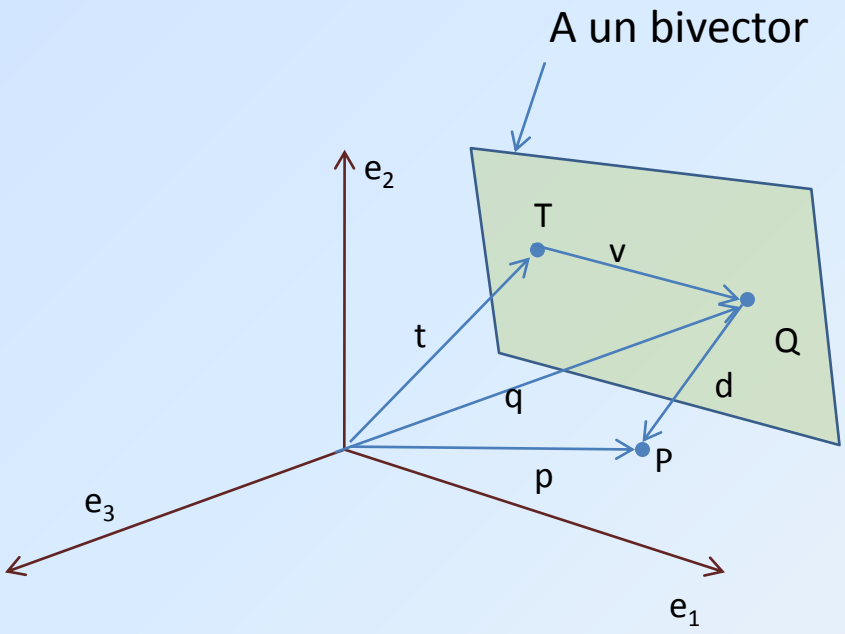
$$Bb = ab^2$$

$$B\frac{b}{b^2} = a$$

$$b^{-1} = \frac{b}{b^2} = \frac{b}{||b||^2}$$

Algebra Geométrica

Corta distancia de un punto a un plano



$$d = p - t - v$$

$$A \wedge v = 0$$

$$Ad = A.d + A \wedge d$$

Como d es perpendicular A

$$Ad = A \wedge d$$

$$A^{-1}Ad = A^{-1}(A \wedge d)$$

$$d = A^{-1}(A \wedge d)$$

$$d = A^{-1}(A \wedge (p - t - v))$$

$$d = A^{-1}(A \wedge (p - t))$$